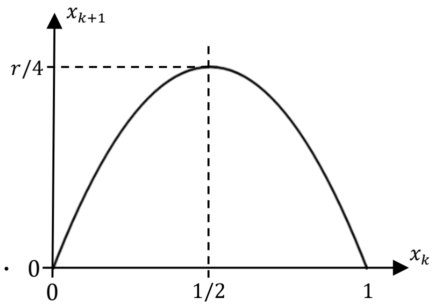
Logistic map



The simplest nonlinear iteration is a quadratic map $x_{k+1} = ax_k^2 + bx_k + c$.
Unlike the tent map and the shift map, it and all its derivatives are continuous.

A simple model with a quadratic map is the **logistic map** $x_{k+1} = rx_k(1 - x_k)$.

Population model

Let y_k be the number of animals in some population for year k. If each animal on average reproduces with a factor r then the next year there will be $y_{k+1} = ry_k$ animals, which would create exponential growth with shortages of food and space. To take this into account let's assume a maximal possible population $\bar{y}$ and that when y_k approaches $\bar{y}$ the population grows in proportion to $(1 - y_k/\bar{y})$. The simplest model that works for both low and high concentrations is $y_{k+1} = ry_k(1 - y_k/\bar{y})$. Divide by $\bar{y}$ and let $x_k = y_k/\bar{y}$ be the population relative to its maximal size, $0 \leq x_k \leq 1$. The resulting model for the population is: $x_{k+1} = rx_k(1 - x_k)$.

Iterator program

To get a feeling for how the iteration behaves you can start the program ITERATOR on the menu. The initial value is usually not of any significance for the long-term behaviour. You can select to give an initial value x_0 or let the program pick a random number from the interval [0,1] for each iteration.

The program offers three alternative iteration maps $x_{k+1} = rf(x_k)$ with the same general unimodal form.
$D_f = [0,1]$, start at $(0,0)$, end in $(1,0)$ and a maximum in between with $0 \leq x_{\max} \leq 1$ so that $\forall k: x_k \in [0,1]$.

Two of the maps has an extra parameter s to adjust the character of the maximum, its power and position.
The fourth map is a sine function, $x_{k+1} = r\sin(\pi x_k)$.

The iteration will, depending on the value of r, either converge towards an attracting k-cycle or move around chaotically, towards a limiting set. To show the attracting set for various r there is a settling-in phase $[0, k_{\min}]$ and a settling-down phase $[k_{\min} + 1, k_{\max}]$. The values of $k_{\min}$ and $k_{\max}$ can be chosen freely in the program.

There are four views, two for the iterations, one for the attracting set for various r and one for the Lyapunov exponent (λ) for the same interval of r with calculation of λ based on the interval $[k_{\min}, k_{\max}]$.

Compositions of the iteration map can be shown in cobweb diagram. Compositions $f^k(x) = f(f^{k-1}(x))$ are interesting to study when the iteration is or getting close to being attracted to a k-cycle.

The iteration can be shown instantaneously or gradually to show how the iteration proceeds and you can zoom in on the attractor diagram, either by giving bounding values directly or by selecting a view with the mouse.

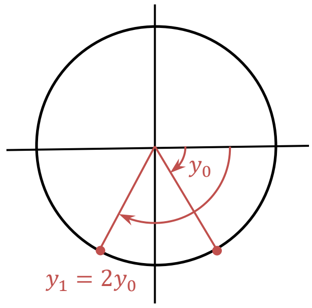
Solutions in closed form

Nonlinear equations do normally not have a solution in closed form, $x_n = g(n)$ with g is expressible as a finite number of operators $(+ - \times \div)$ and elementary functions like roots, exponents and trigonometric functions.
The only way to get the solutions is by numerical methods and calculate $x_1, x_2, \ldots, x_{n-1}$ and then x_n.
The situation is similar for nonlinear differential equations.

An exception to this rule is the logistic map when $r = 4$: $x_{k+1} = 4x_k(1 - x_k)$ and $r = \pm 2$.

Bernoulli map:
$y_{n+1} = 2y_n \bmod 1$

A change of variables, $x_k = 1/2 \cdot (1 - \cos 2\pi y_n)$ gives:



$$1/2 \cdot (1 - \cos(2\pi y_{n+1})) = (1 - \cos(2\pi y_n)\,(1 + \cos(2\pi y_n))$$

$$1/2 \cdot (1 - \cos(2\pi y_{n+1})) = 1/2 \cdot (1 - \cos^2(2\pi y_n)) \qquad \left[\cos^2\alpha = \frac{\cos 2\alpha + 1}{2}\right]$$

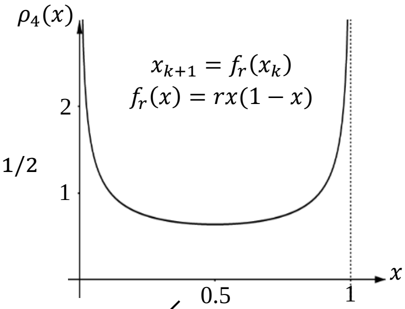
$$1/2 \cdot (1 - \cos(2\pi y_{n+1})) = 1/2 \cdot (1 - \cos(4\pi y_n)) \;\rightarrow\; y_{n+1} = 2y_n \bmod 1$$

$$y_n = 2^n y_0 \bmod 1$$

$$x_n = 1/2 \cdot (1 - \cos(2\pi 2^n y_0)) \;\text{ with }\; y_0 = \frac{1}{2\pi}\cos^{-1}(1 - 2x_0)$$

on a circle with $r = 1/2\pi$ and $C = 1$

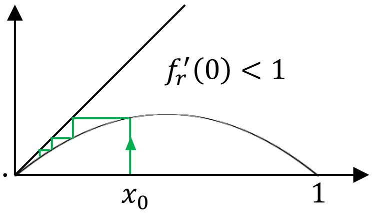
The invariant density $\rho(x)$ i.e. the distribution of x_k for a typical iteration series of $x_k = 4x_k(1 - x_k)$ can be calculated based on the Bernoulli map that has a uniform distribution of iterates for a random initial value, $\tilde{\rho}(y) = 1$

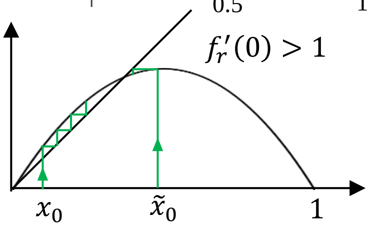


$$\rho(x)|dx| = \tilde{\rho}(y)|dy| \;\text{ and }\; x = 0.5 \cdot (1 - \cos 2\pi y) \quad\rightarrow\quad \rho(x) = \pi^{-1}\big(x(1-x)\big)^{-1/2}$$

Iterations, fixpoints and cycles

There are two parameters to vary, x_0 and r when studying the iterates $x_0, x_1, x_2, \ldots$ of the logistic map, $x_{k+1} = rx_k(1 - x_k) = f_r(x_k)$. Start from $r = 0$ and go up to 4, for each r study the effects of different x_0.





From the diagrams it's clear that when $f_r'(0) < 1$ there is one fixpoint $x^* = 0$ that is attractive (stable). The population will go extinct no matter the initial value of x_0.

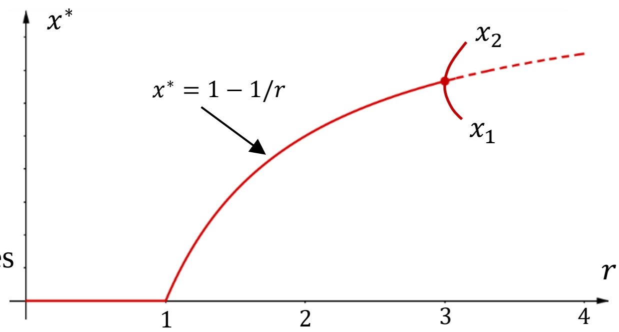
$\lim_{k\to\infty} x_k = 0$ for every $x_0 \in D_f = [0,1]$ whenever $f_r'(0) < 1$ $\qquad f_r'(x) = r(1 - 2x) \rightarrow f_r'(0) < 1$ if $r < 1$

The attracting area of a stable fixpoint is called the **basin of attraction**.

As r passes 1 the fixpoint x^* goes from attracting to repelling and a new fixpoint emerges.

$f_r(x^*) = x^* \;\rightarrow\; rx^*(1 - x^*) = x^* \;\rightarrow\; x^* = 1 - 1/r$

$|f_r'(1 - r^{-1})| = |2 - r|$



The new fixpoint is stable $|f_r'(x^*)| < 1$ when $1 < r < 3$ and has a basin of attraction $0 < x < 1$. It becomes unstable when $|f_r'(x^*)| > 1$, i.e. when $r > 3$ and things get interesting. The fixpoint becomes repelling and an attractive 2-cycle emerges with two points that satisfy $f\big(f(x^*)\big) = x^*$ and $|Df^2(x^*)| < 1$.

An n-cycle has n distinct points $x_1, x_2, \ldots x_n$ with $f^n(x_i) = x_i$ and $f^k(x_i) \neq x_i$ for $k < n$.
The cycle is attracting if $|Df^n(x^*)| < 1$ and repelling if $|Df^n(x^*)| < 1$.
$Df^n(x^*) = f'(x_1) \cdot f'(x_2) \cdot \ldots \cdot f'(x_n) \rightarrow f'(x_i)$ has the same value for all x_i.
The approach or retreat from a cycle is oscillating or one-sided depending on the sign of $Df^n(x^*)$.

Solving $f^2(x^*) = x^* \;\rightarrow\; x^* = \dfrac{r + 1 \pm \sqrt{r^2 - 2r - 3}}{2r}$ an attractive cycle of 2 points for $3 < \mathrm{r} < 3.4495$

Analyzing $f^n(x) = x \;\rightarrow$ $\quad 2^2$ cycle: $r \in 3.4495 \rightarrow 3.5441$ $\quad r_2 = 3 \;\; r_{2^2} = 3.4495 \;\; r_{2^3} = 3.5441$ *etc.*
2^3 cycle: $r \in 3.5441 \rightarrow 3.5644$ $\quad$ with accumulation point $r_{2^\infty} = 3.5699\ldots$

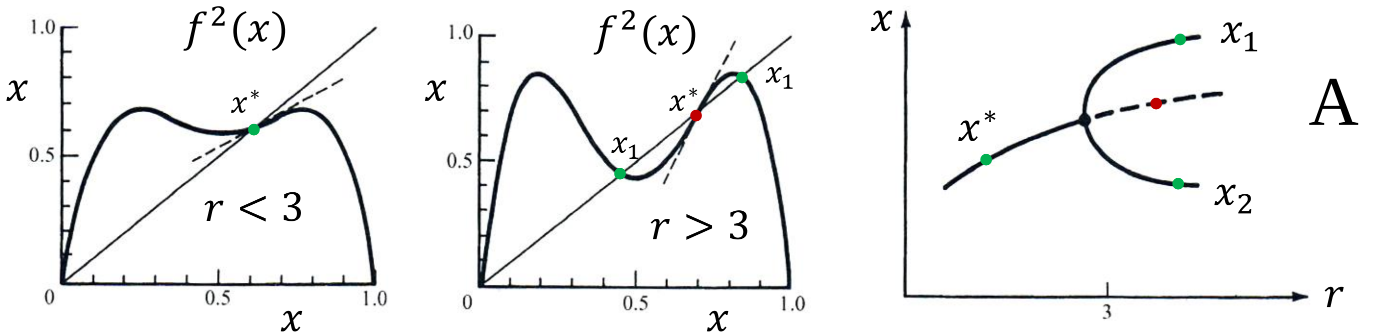
The process of **period doubling** is called **bifurcation**.

Bifurcation

Type of fixpoint	$f'(x^*)$
unstable, monotonous	$1 \to \infty$
hyperbolic, saddle point	1
stable, monotonous	$0 \to 1$
super-stable	0
stable, alternating	$-1 \to 0$
hyperbolic, saddle point	-1
unstable, alternating	$-\infty \to -1$

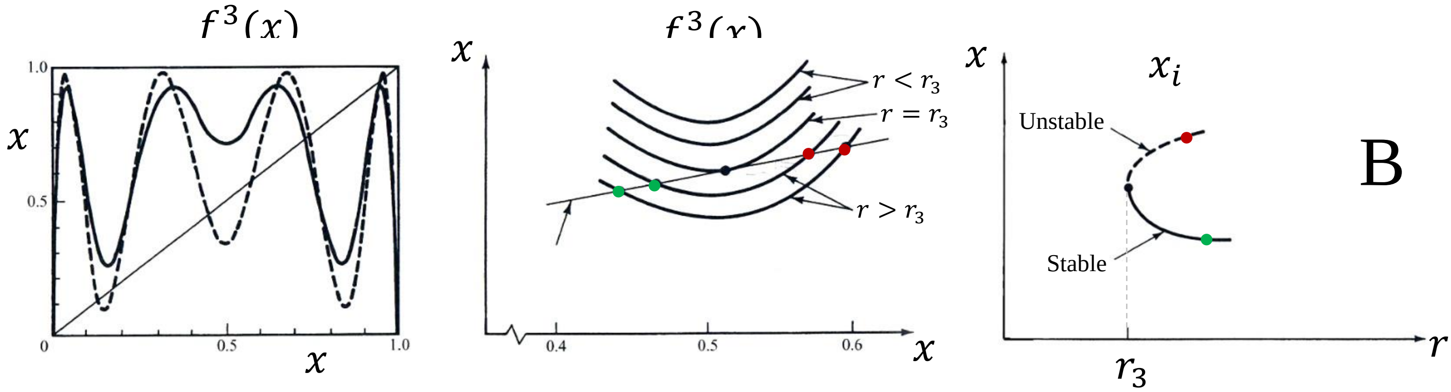
The fixpoint $x^* = 1 - 1/r$ of $f(x)$is also a fixpoint of $f^2(x)$.
As r passes through 3, $f'(x^*) = 2 - r$ passes through -1,
from being a stable fixpoint to becoming unstable.

As this happens the slope of f^2 with $Df^2(x^*) = \left(f'(x^*)\right)^2$
goes from below 1 to above 1, two new fixpoints of f^2 emerge.
These new fixpoints $x^* = x_{1,2}$ are not fixpoints of f.
$f^2(x^*) = x^*$ and $f(x^*) \neq x^* \quad \Rightarrow \quad$ They belong to a period 2 orbit.
Their slope $Df^2(x^*) = f'(x_1)f'(x_2)$ is such that the orbit is stable.
The process is a general feature called a **period doubling bifurcation** that happens repeatedly as stable cycles becomes unstable when r is increased and fixpoint splits into two and create new orbits of double period.



An illustration of the bifurcation process is in this Geogebra animation that shows f, f^2, f^3, f^4 and fixpoints as r is animated from $r = 0$ to $r = 4$. It shows how orbits of period 2^n are created. One can also see another type of process, a process that is associated with f^3, the creation of a stable orbit of period 3.

As r is increased $f^3(x)$ reaches the line $y = x$ at three points simultaneously $x^* = x_{1,2,3}$. This is not a miracle since $f^3(x_1) = x_1 \Rightarrow f^3(x_2) = x_2 \Rightarrow f^3(x_3)$ where x_1, x_2, x_3 are a sequence in the iteration of f.
They all have same slope in f^3 since $Df^3(x^*) = f'(x_1) \cdot f'(x_2) \cdot f'(x_3)$ and when they appear $Df^3(x^*) = 1$.



The first period-3 orbit happens when $\begin{cases} f_r^3(x) = x \\ Df_r^3(x) = 1 \end{cases} \to r_3 = 3.8284 \ldots \ \to r_{3\cdot 2} = 3.8414 \ldots \ \to \cdots$

The period-3 orbit is followed by bifurcation in the same fashion as above, generating orbits of period $3 \cdot 2^n$.
The intervals of these orbits get shorter and shorter and end up in an accumulation point at $r_{3\cdot 2^\infty} = 3.8540 \ldots$

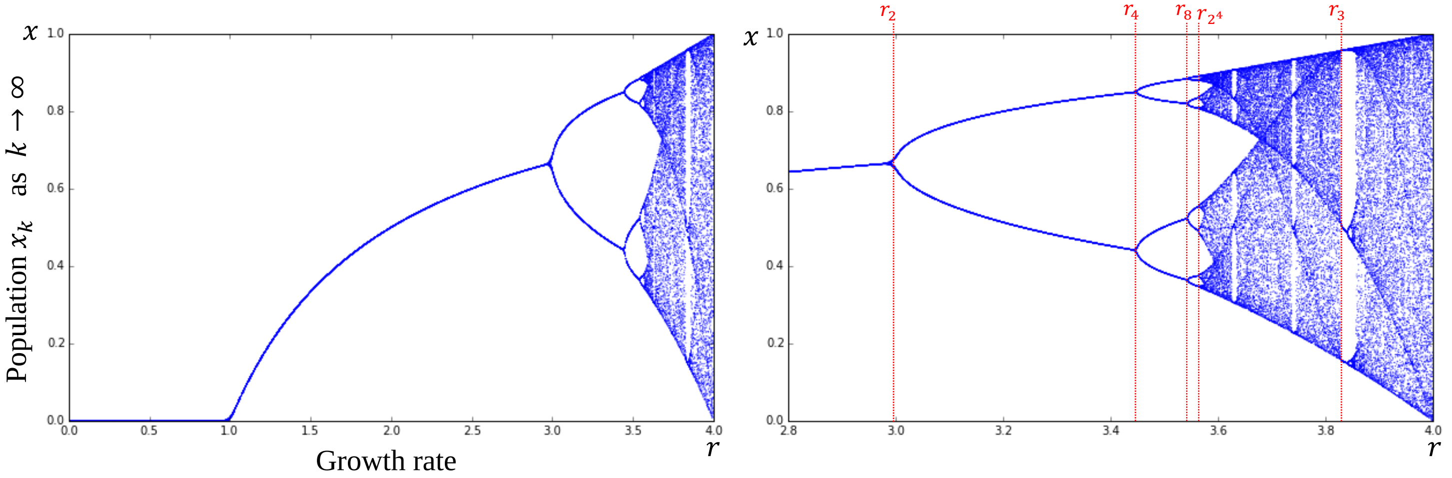
In a similar fashion there are period-p orbits followed by bifurcations $p \cdot 2^n$ and accumulation points $p \cdot 2^\infty$.
The two different types of split (A and B) are called **pitchfork bifurcation** (A) and **tangent bifurcation** (B).

Where does the period-3 orbit come from and what happens after the accumulation points r_{2^∞} and $r_{3\cdot 2^\infty}$?

Bifurcation diagram and Chaos

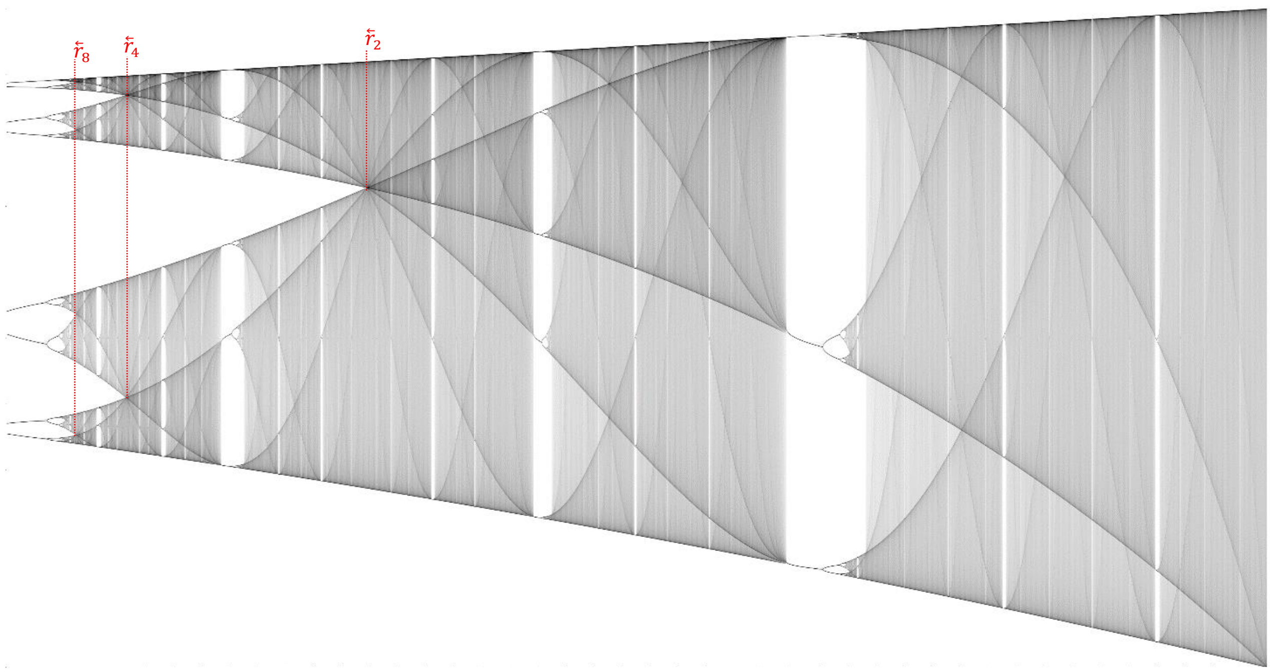
Too see what happens after the accumulation point r_{2^∞} and before the period-3 orbit r_3 one uses a plot of the 'attracting' set as a function of r, from $r = 0$ to $r = 4$. For each r, take a random x_0 or $x_0 = 1/2$, which one doesn't matter and iterate a few hundred times ($k_{\min}$) until the iterates $x_{k+1} = rx_k(1 - x_k)$ are close to their 'attracting set' and then iterate a few hundred times more ($k_{\max} - k_{\min}$) to get a picture of what the iteration is approaching. $k_{\min}$ and $k_{\max}$ are parameters you chose in the ITERATOR program.

Bifurcation diagram with enlargement



The diagrams show the initial attracting point x^*, first $x^* = 0$ and later $x^* = 1 - r^{-1}$ which splits at $r_2 = 3$ to $x_{1,2} = 1/2 \cdot \left(1 + r^{-1} \pm \sqrt{r^2 - 2r - 3}\right)$ followed by a series of bifurcations at r_{2^n} ending in $r_{2^\infty} = 3.5699 \ldots$ Then comes periods of chaos where the iterates fluctuate wildly inside one or several disconnected intervals. The r-values with chaotic orbits are interspersed with intervals of periodic orbits. The largest is the period-3 window starting at r_3. First when $r = 4$ does the iteration fluctuate over the whole interval $[0,1]$ but not uniformly. The distribution of iterates x_k follows the invariant density $\rho_4(x) = \frac{1}{\pi\sqrt{x(1-x)}}$.

As r decrease from $r = 4$ the x-area that the x_k occupy when there is chaos goes from one connected interval to 2 and then 4 and after a series of splits $\overleftarrow{r}_2, \overleftarrow{r}_4, \overleftarrow{r}_8, \ldots$ there is an accumulation point $\overleftarrow{r}_{2^\infty}$ that coincides with r_{2^∞}.



The bifurcation diagram shows several bands with periodic orbits that emerge from chaos. Each time when an orbit of period p appears there is a series of bifurcations with orbits of period $p \cdot 2^n$.. Each band in a period-p window contains a miniature version of the whole diagram. The bifurcation diagram is self-similar with smaller copies of itself within smaller copies of itself in all eternity.

When the chaotic x-region is split in several bands the iterates alternates between the bands in the same order as in the corresponding finite orbit $(r_{2^n} \leftrightarrow \overleftarrow{r}_{2^n})$. Eventually the iterates comes arbitrarily close to every point in the band.

The periodic windows are dense throughout the chaotic range $[r_{2^\infty}, 4]$, i.e. in any neighbourhood $[r-\varepsilon, r+\varepsilon]$ there will always be some periodic orbit of type $p \cdot 2^n$. The number of windows of period p is $(2^p - 2)/(2p)$. The probability of finding a chaotic orbit in $[r_{2^\infty}, 4]$ has been shown to be bigger then zero (as the Lebesgue measure of the set of chaotic $r's$), something resembling a fat Cantor set where you start from [0,1] and instead of repeatedly removing a middle third of remaining intervals you remove a smaller proportion each time.

Three theorems on orbit structure

There are three famous theorems about the orbit structure of unimodal maps:

- Sharkovskii's theorem on existence of periodic orbits (1964)
- Metropolis, Stein & Stein's theorem on the organization of periodic orbits (1973)
- Li & Yorke's theorem that a period-3 orbit implies existence of chaos (1975)

Sharkovskii's theorem: There is an ordering of $\mathbb{Z}^+$ such that if a continuous map $f: \mathbb{R} \to \mathbb{R}$ has a periodic orbit of length n, $f^n(x^*) = x^*$, then f will also have a period-p orbit for every p that is after n in the order.

Sharkovskii's order of the positive integers is as follows:

$$\underbrace{3 \to 5 \to 7 \to \cdots}_{\text{odd numbers}} \to \underbrace{2\cdot 3 \to 2\cdot 5 \to 2\cdot 7 \to \cdots}_{2^1\cdot\text{odd number}} \to \underbrace{2^2 3 \to 2^2 5 \to 2^2 7 \to \cdots}_{2^2\cdot\text{odd number}} \ \ldots \underbrace{\to 2^3 \to 2^2 \to 2 \to 1}_{\text{no odd factor}}$$

Ex: If $f_r(x) = rx(x-1)$ has a period-8 orbit for some initial value $x_{0,8}$ then there will be other initial values $x_{0,4}$, $x_{0,2}$, and $x_{0,1}$ with orbits of period 4, 2 and 1 respectively for the same r.

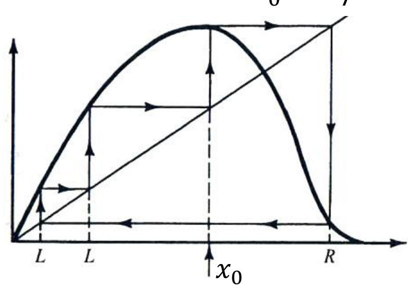
The animation of f, f^2, f^3, f^4 as r increase shows that there is an orbit of period-3 for every $r \in [r_3, 4]$. This means that there must also be initial values $x_{0,n}$ that will result in orbits of any period n besides 3.

Note that ordering is total but not a well-order, the subset $\{1,2,4,8 \ldots\}$ does not have a least element. The theorem doesn't concern the stability of the orbits. For a proof see [24], in the online reference list.

The second theorem by Metropolis et al. concern the structural universality of super-stable orbits of unimodal maps of the form $x_{k+1} = rf(x_k)$. An orbit is super-stable if its stability coefficient $f'(x^*)$ is zero, which means a Lyapunov exponent $\lambda(x^*) = -\infty$.

For the logistic map with max at $x = 1/2$ all super-stable orbits pass $x = 1/2$ so we can assume $x_0 = 1/2$.

A p-orbit $x_0, x_1, x_2, \ldots, x_{p-1}$ is categorized by a sequence of p - 1 letters. Letter k is L if x_k goes to left, $x_k < x_0$ and R if x_k goes to the right, $x_k > x_0$.



As r is increased there will be a sequence of orbits labelled by their RL-sequences, the orbit on display will be categorized as RLL.

Table of all super-stable orbits up to period 6 ordered by r for the logistic map and for the sine map:

Period	**Sequence**	**Parameter r of $x_{k+1} = rx_k(1-x_k)$**	**Parameter r of $x_{k+1} = r\sin \pi x_k$**
2	R	3.2360680	0.777338
4	RLR	3.4985617	0.8463822
6	RLR^3	3.6275575	0.8811406
5	RLR^2	3.7389149	0.9109230
3	RL	3.8318741	0.9390431
6	RL^2RL	3.8445688	0.9435875
5	RL^2R	3.9057065	0.9633656
6	RL^2R^2	3.9375364	0.9735656
4	RL^2	3.9602701	0.9820353
6	RL^3R	3.9777664	0.9892022
5	RL^3	3.9902670	0.9944717
6	RL^4	3.9975831	0.9982647

Theorem of Metropolis, Stein & Stein:
The ordering of RL-strings for super-stable orbits is universal for all maps on the interval [0,1] with a differentiable maximum that fall of monotonically on both sides.

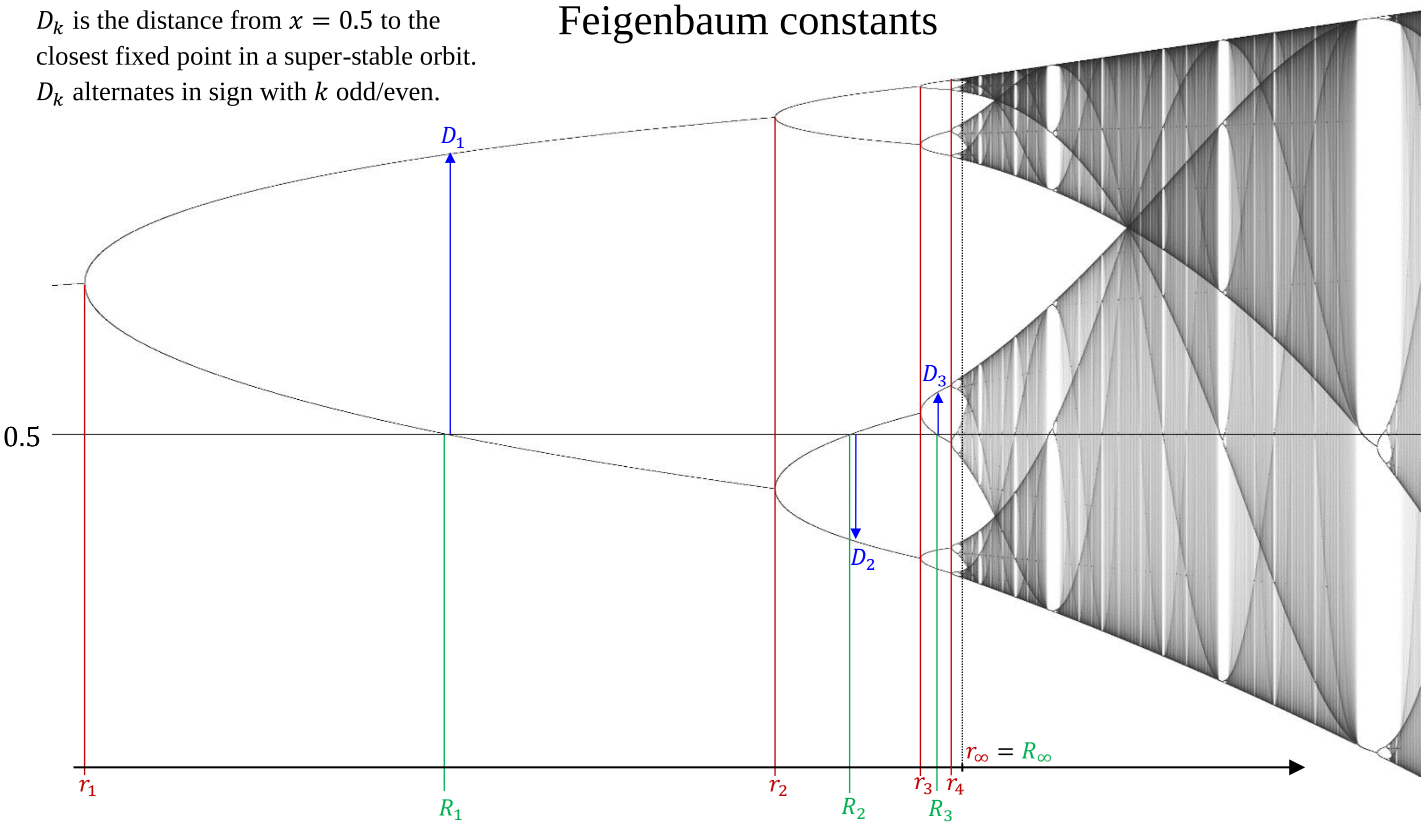
Note that the universality does not depend on the order of the maximum.

Li & Yorke's theorem is also known as 'period three implies chaos'. It is is an extension of the Sharkovskii theorem. They showed that the existence of a period-3 orbit implies the existence of an uncountable set of orbits that never settle into a periodic cycle. They introduced the term *Chaos* for this situation.

A consequence is that for a given value of r in $[r_3, r_6]$ where there is an attracting period-3 orbit there will also be an uncountable number of initial values x_0 with non-periodic, non-attracting and complicated orbits. These initial orbits turn out to be of zero Lebesgue measure.

D_k is the distance from $x = 0.5$ to the closest fixed point in a super-stable orbit. D_k alternates in sign with k odd/even.

Feigenbaum constants



There is a set of constants that are a lot more general than the r_k, R_k, D_k that you see in the bifurcation diagram. Those constants are tied to the specific form of the logistic map $f_r(x) = rx(1-x)$. The new class of constants are not as general as the three theorems that apply to a very broad class of maps with a single peak but still very general and of big interest, since they also appear in many natural phenomena in physics and elsewhere.

The intervals $\Delta r_k = r_{k+1} - r_k$, ΔR_k and ΔD_k shorten with k and goes to zero as $k \to \infty$, a guess is that they might approach geometric series with asymptotic scaling factors δ^{-1} and α^{-1} where:

$$\delta_r \equiv \lim_{k\to\infty} \frac{\Delta r_k}{\Delta r_{k+1}} \quad \delta_R \equiv \lim_{k\to\infty} \frac{\Delta R_k}{\Delta R_{k+1}} \quad \alpha \equiv \lim_{k\to\infty} \frac{-D_k}{D_{k+1}} \quad (-D_k \to \alpha > 0)$$

The idea is correct and $\delta_r = \delta_R$. Numerical computations are easier to do with R_k than r_k. For $n \gg 1$ we get:

$r_n \approx r_\infty - C \cdot \delta^{-n}$
$R_n \approx R_\infty - C' \cdot \delta^{-n}$ For a constant C and a different constant C' ($R_\infty = r_\infty$).

If the constants would be different for each iteration map, they would not be very interesting but it turns out that they are the same for every iteration map with a quadratic maximum, $f'(x_c) = 0$ and $f''(x_c) < 0$ at the peak.

α and δ are called Feigenbaums constants: $\delta = 4.6692016091 \ldots$
$\alpha = 2.5029078750 \ldots$

Orbits with period p ($p = 3,5,6, \ldots$) that start with a tangent bifurcation from a chaotic r-region followed by a series of bifurcations giving rise to $p \cdot 2^n$- orbits give corresponding series in r and R.
They have the same asymptotic scaling but with different constants, $r_{p,n} \approx r_{p,\infty} - C_p \cdot \delta^{-n}$ for large n.

Functions with other types of maxima have their own constants, the generalized Feigenbaum constants.
Consider the iteration map $x_{k+1} = 1 - \tilde{r}|x_k|^q$ where $1 < \tilde{r} < 2$ and $q > 1$.
The maximum is cubic if $q = 3$ and quartic if $q = 4$ etc. $\delta(q)$ increases with q and $\alpha(q)$ decreases with q.

$q \to$	3	4	5	6
$\delta(q)$	5.967968…	7.284686…	8.349499…	9.296246…
$\alpha(q)$	1.927690…	1.690302…	1.555771…	1.467742…

Limiting values:

$\lim_{q\to\infty} \delta(q) = 29.57$ $\lim_{q\to\infty} \alpha(q)^{-q} = 0.033$

$\lim_{q\to 1+} \delta(q) = 2$ $\lim_{q\to 1+} \alpha(q) = \infty$

Some books in the reference list mention period triplings $p \cdot 3^n$ and higher that occur at $\overline{r} = \overline{r}_\infty - C' \cdot \delta'^{-n}$ with different Feigenbaum constants δ'. The period triplings are said to occur with $\delta' = 55.247$, but there are no trifurcations where an attracting fixpoint branch splits into three branches like the trident of Poseidon, not in the logistic map or any other iteration map. What do they mean?

There is a period-3 orbit that arise from ‘chaos’. When the bifurcation process is finished each branch will give rise to a miniature version of the bifurcation diagram, with new period-3 orbits. From the global view this is a period-3^2orbit. The self-similarity of the diagram means that this will repeat indefinitely. Is this the answer?

Experimentation with the ITERATOR gives $r_3 = 3.8285 \to$ $r_{3\cdot 3} = 3.8536 \to r_{3\cdot 3^2} = 3.85407$
The first approximation of δ_3 becomes $(3.8536 - 3.8282)/(3.85407 - 3.8536) \approx 53.4$

Is it a correct assumption?

The assumption is confirmed by data from Zeng that gives $\tilde{R}$-values for super-stable orbits from various positions in the bifurcation diagram. The data is based on the related map $x_{k+1} = 1 - \tilde{r}x_k^q$.
There is a correspondence with this map when $q = 2$ and the logistic map $x_{k+1} = rx_k(1 - x_k)$.
The translation between r and $\tilde{r}$ is $\tilde{r} = r(r-2)/4$ and similarly for R and $\tilde{R}$.

The following table is based on data for super-stable orbits from the article by Zeng et al..

RL-Sequence	n	Period	R	$\tilde{R}$ $(q=2)$	δ	$\tilde{R}$ $(q=4)$	δ	$\tilde{R}$ $(q=6)$	δ
$(RL)^{*n}$	1	3	3.831874	1.754877	55.247	1.856674	85.81	1.898653	130.3
	2	9	3.853675	1.785865		1.908694		1.948454	
	3	27	3.854070	1.786429		1.909328		1.948863	
	4	81	3.854078	1.786440		1.909335		1.948866	
$(RL^2)^{*n}$	1	4	3.960270	1.940799	981.6	1.981964	1275	1.991457	2220
	2	16	3.961555	1.942702		1.985501		1.994205	
$(RLR^2)^{*n}$	1	5	3.738914	1.625413	255.5	1.725711	291.9	1.776745	431.9
$(RL^2R)^{*n}$	1	5	3.906697	1.862222	1287	1.942075	1418	1.967639	2434
$(RL^3)^{*n}$	1	5	3.990267	1.985424	17000?	1.997746	21000?	1.999287	43000?

Remains to explain what exponentiation of RL-sequences with the operation * means.

The composition operator * on two RL-sequences P and Q where Q = $\sigma_1\sigma_2 \ldots \sigma_{q-1}$ $(\sigma_k \in \{R, L\})$ depends on the number of Rs in Q.

If the number of R symbols in Q is even: P*Q = Pσ_1Pσ_2P…Pσ_{q-1}P
If the number of R symbols in Q is odd: P*Q = Pτ_1Pτ_2P…Pτ_{q-1}P $\tau_i \neq \sigma_i$ (the opposite symbol)

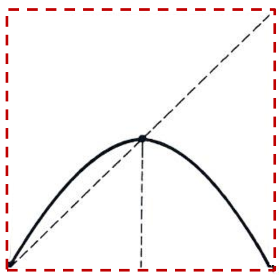
The operation is associative but not commutative. If P is an orbit of length p and Q an orbit of length q then PQ represents an orbit of length $p(p-1) + q$ and $P^{*n} = \underbrace{P * P * \ldots * P}_{n\text{ times}}$ represents a p^n-orbit.

For example, some super-stable orbits with periods 2^n and 3^n of the logistic map are as follows:

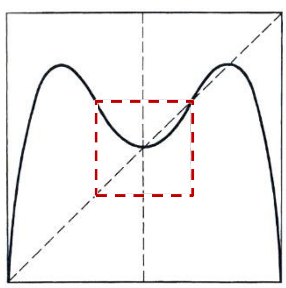
RL-Sequence	Period	R	$x_0, x_1, x_2, \ldots$
R	2	3.23607..	½→0.809..→½
$R^{*2} = RLR$	4	3.49856..	½→0.874..→0.383..→0.827..→½
$R^{*3} = RLRRRLR$	8	3.55464..	½→0.888.. →0.351.. →0.810.. →0.545.. →0.881.. →0.372.. →0.830.. →½
RL	3	3.831874..	½→0.957.. →0.154.. →½
$(RL)^{*2} = RLLRLRRL$	9	3.853675..	½→0.963..→0.135..→0.452..→0.954..→ 0.166..→ 0.535..→ 0.958..→ 0.153..→½

Numerical experimentation is good but a secure mathematical foundation of Feigenbaum constants is needed.

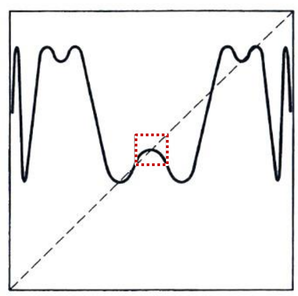
Let R_n be the r-points where $x_{k+1} = rx(1-x) = f(x, r)$ har super-stable orbits of period 2^n, $f'\left(\frac{1}{2}, R_n\right) = 0$.



$f(x, R_0)$



$f^2(x, R_1)$



$f^4(x, R_2)$

The portions of the graphs inside the dashed squares look similar apart from reflections in the point $\left(\frac{1}{2},\frac{1}{2}\right)$.
The square side reduces with a scale factor $\hat{\alpha}_n$ that appears to have a limit $\hat{\alpha}$.

With a critical point in $x = 0$ like $f(x,\tilde{r}) = 1 - \tilde{r}|x|^q$ instead of $x = 1/2$ this is expressed by:
$f^{2^n}(x, R_n) \cong (-\alpha_n) f^{2^{n+1}}(-x/\alpha_n, R_{n+1})$
The similarity assumption is strengthened by the fact that the limit as $n \to \infty$ exists.
$\lim_{n\to\infty} (-\alpha_n)^n f^{2^n}(x/(-\alpha_n)^n, R_n) = \varphi\,(x)$

This limit satisfies the **Cvitanovic-Feigenbaum functional equation**: $T[\varphi(x)] = \varphi(x)$
with a functional operator T that acts on a functional space, a Banach space Ω.
$\varphi(x)$ is a fixpoint of the **period-doubling operator** $T\colon \Omega \to \Omega$,
or rather T_q if we deal with functions of the form $f(x) = 1 + F(|x|^q)$, such as $f(x) = 1 + \tilde{r}x^2$.

$T[\varphi](x) = -\alpha\varphi\left(\varphi\left(-\frac{x}{\alpha}\right)\right)$ If φ is a solution so is $k^{-1}\varphi(kx)$, this is fixed by demanding $\varphi(0) = 1$.

An approximate solution $\varphi(x) = 1 - ax^2 + O(x^4) \;\; \to \;\; 1 - ax^2 \cong -\alpha(1-a) - (2a^2/\alpha)x^2 \;\; \to$
$\alpha = 1 + \sqrt{3} = 2.73$ ... a first approximation of $\alpha = 2.502$ ... found by Feigenbaum.

On a Banach manifold you can do differential calculus. To get at δ, you need to analyse the local linearization, the Fréchet derivative of T_q at the fixed point $\varphi(x)$.

$$L_q[\Psi](x) = -\alpha\{\varphi'\big(\varphi(-x/\alpha)\big)\cdot\Psi(-x/\alpha) + \Psi\big(\varphi(-x/\alpha)\big) + \Psi(1)[\varphi'(x)\cdot x - \varphi(x)]\}$$

The Feigenbaum constants $\varphi(q)$ are given as the largest eigenvalue of L_q,
which is the only eigenvalue of L_q outside the unit disc.

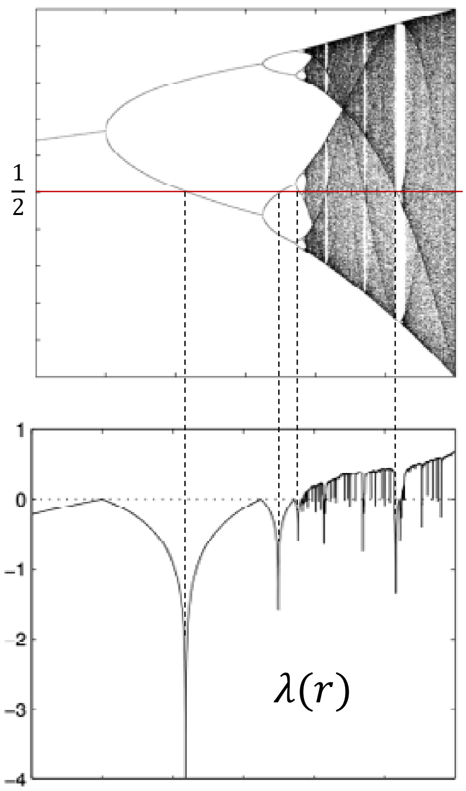
Another functional equation is needed for period triplings: $T_3[\varphi](x) = -\alpha\left(\varphi\left(\varphi\big(\varphi(-x/\alpha)\big)\right)\right)$.

Lyapunov exponents and Invariant densities

The first sign of chaos is sensitive dependence on initial conditions which is measured by a positive Lyapunov exponent $\lambda(x_0)$.
The Lyapunov exponent measures exponential separation $\delta \to \delta e^{n\lambda}$
After n iterations $\delta e^{n\lambda(x_0)} \approx |f^n(x_0+\delta) - f^n(x_0)|$.

$$\lambda_r(x_0) = \lim_{n\to\infty}\frac{1}{n}|Df_r^n(x_0)| = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\ln|f_r'(x_k)|$$

At super-stable orbits with $f_r'(x^*) = 0$, λ goes to minus infinity.



The distribution of x_k in a chaos-regions is not as smooth for $r < 4$ as it is for $r = 4$

$$\rho_4(x) \propto \frac{1}{\sqrt{x(1-x)}}$$

$\rho(x)$ is the invariant density for the distribution of x_k.